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A REFINEMENT OF YOUNG'S INEQUALITY

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Abstract. We present an improved version of Young's inequality as well as an operator inequality version of it. Our result is compared to the latest refinements.

1. Introduction

Throughout this paper let a, b be arbitrary positive numbers and $0 \le p \le 1$. Young's inequality or p-weighted arithmetic-geometric mean inequality says

$$a^p b^{1-p} \le pa + (1-p)b.$$

During the past years, several refinements were given for Young's inequality, see for example [1].

In [2], the following inequality was proved by S. Furuichi:

$$S\Big(\Big(\frac{b}{a}\Big)^r\Big) a^p b^{1-p} \le pa + (1-p)b, \tag{1.1}$$

where $r = \min\{p, 1-p\}$ and $S(x)$ is Specht's ratio (see [3]).

In [5], it was seen that

$$K^r\Big(\frac{b}{a}\Big) a^p b^{1-p} \le pa + (1-p)b, \tag{1.2}$$

where $r = \min\{p, 1-p\}$ and $K(x) = \frac{(1+x)^2}{4x}$ is Kantorovich's constant. It was also proved that $S(x^s) \le K^s(x)$ for $x > 0$, $0 \le s \le 1/2$, which means that (1.1) is a consequence of (1.2).

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In the special case when $p = 1/2$, Young's inequality is the well-known arithmetic-geometric mean inequality

$$\sqrt{ab} \le \frac{a+b}{2}.$$

In [4], an improved version of the arithmetic-geometric mean inequality was proved by L. Zou and Y. Jiang:

$$\Big(1 + \frac{(\log a - \log b)^2}{8}\Big)\sqrt{ab} \le \frac{a+b}{2}, \tag{1.3}$$

moreover, the relationship between $S\big(\sqrt{b/a}\big)$ and $1 + \frac{(\log a - \log b)^2}{8}$ was discussed. While inequality (1.3) was proved in [4], we now give an alternative proof, which inspired the proof of our main result presented later.

Proof of (1.3). Without loss of generality, we can assume that $a \ge b$. Then (1.3) is equivalent to

$$\Big(1 + \frac{(\log a - \log b)^2}{8}\Big)\sqrt{\frac{a}{b}} \le \frac{\frac{a}{b}+1}{2}.$$

Using a substitution $x = \frac{a}{b} \ge 1$, we need that

$$\Big(1 + \frac{\log^2 x}{8}\Big)\sqrt{x} \le \frac{x+1}{2},$$

which is equivalent to

$$\frac{\log^2 x}{4} \le \frac{x - 2\sqrt{x} + 1}{\sqrt{x}}.$$

Now substituting $y = \sqrt{x} \ge 1$, the needed relation is

$$\log^2 y \le \frac{(y-1)^2}{y},$$

which is equivalent to

$$\log y \le \frac{(y-1)}{\sqrt{y}}.$$

This relation was proved in [2, Lemma 2]. $\square$

However, the relationship between $K^{1/2}(\frac{b}{a})$ and $1 + \frac{(\log a - \log b)^2}{8}$ was not discussed in [4]. Now we prove that

$$K^{1/2}(x) \ge 1 + \frac{\log^2 x}{8} \quad \text{for } x > 0, \tag{1.4}$$

which means that we can obtain (1.3) from (1.2).

PROOF OF (1.4). Let

$$f(x) := \frac{1+x}{2\sqrt{x}} - 1 - \frac{\log^2 x}{8}\,.$$

We need to see that $f(x) \geq 0$ on $(0,\infty)$. Since $f(1) = 0$ and $f(x)$ is differentiable on $(0,\infty)$, it is enough to see that $f'(x) \leq 0$ on $(0,1)$ and $f'(x) \geq 0$ on $(1,\infty)$. As

$$f'(x) = \frac{x-1}{4x\sqrt{x}} - \frac{\log x}{4x}\,,$$

it is enough to see that the derivative of

$$g(x) := 4xf'(x) = \sqrt{x} - \frac{1}{\sqrt{x}} - \log x$$

is non-negative, that is

$$g'(x) = \frac{x - 2\sqrt{x} + 1}{2x\sqrt{x}} \geq 0 \quad \text{for } x > 0,$$

which inequality holds. $\square$

2. Main results

We prove an inequality similar to (1.1) and (1.2), meanwhile we generalize (1.3). That is,

$$\left(1 + Q(p)(\log a - \log b)^2\right) a^p b^{1-p} \leq pa + (1-p)b, \tag{2.1}$$

where $Q(p) = \frac{p^2}{2}\left(\frac{1-p}{p}\right)^{2p}$ for $0 < p < 1$ and $Q(0) = Q(1) = 0$.

PROOF OF (2.1). We can suppose that $0 < p < 1$. A division by b and a substitution $x = \frac{a}{b}$ imply that we need to see

$$\left(1 + Q(p)\log^2 x\right) x^p \leq px + 1 - p.$$

Hence if we set

$$f(x) := px^{1-p} + (1-p)x^{-p} - 1 - Q(p)\log^2 x,$$

then it is enough to prove that $f(x) \geq 0$ on $(0,\infty)$. Since $f(1)=0$ and $f(x)$ is differentiable on $(0,\infty)$, it is enough to see that $f'(x) \leq 0$ on $(0,1)$ and $f'(x) \geq 0$ on $(1,\infty)$. As

$$f'(x) = \frac{p(1-p)(x^{1-p}-x^{-p})}{x} - \frac{2Q(p)\log x}{x},$$

after substituting $y = x^p > 0$, it is enough to see that the derivative of

$$g(y) := xf'(x) = p(1-p)\Big(y^{\frac{1}{p}-1} - \frac{1}{y}\Big) - \frac{2}{p}\,Q(p)\log y$$

is non-negative, that is

$$g'(y) = \frac{(1-p)^2 y^{1/p} - p(\frac{1-p}{p})^{2p} y + p(1-p)}{y^2} \geq 0 \quad \text{for } y > 0.$$

Hence we need the non-negativity of

$$h(y) := y^2 g'(y) = (1-p)^2 y^{1/p} - p\Big(\frac{1-p}{p}\Big)^{2p} y + p(1-p).$$

We can obtain the required result from the facts

$$h'(y) = \frac{(1-p)^2}{p} y^{\frac{1}{p}-1} - p\Big(\frac{1-p}{p}\Big)^{2p}, \quad h''(y) = \frac{(1-p)^3}{p^2} y^{\frac{1}{p}-2} > 0,$$
$$h'\Big(\Big(\frac{p}{1-p}\Big)^{2p}\Big) = h\Big(\Big(\frac{p}{1-p}\Big)^{2p}\Big) = 0,$$

which altogether mean that $h(y)$ is convex and its minimum is 0. $\square$

We remark that in formula (2.1), $Q(p) = Q(1-p)$ as

$$Q(1-p) = \frac{(1-p)^2}{2}\Big(\frac{p}{1-p}\Big)^{2-2p} = \frac{p^2}{2}\Big(\frac{1-p}{p}\Big)^{2p} = Q(p).$$

We can also draw the following consequence of (2.1):

$$\big(1 + R(p)(\log a - \log b)^2\big)\, a^p b^{1-p} \leq pa + (1-p)b,$$

where $R(p) = \min\big\{\frac{p^2}{2}, \frac{(1-p)^2}{2}\big\}$. This can be easily obtained from the relation $Q(p) \geq R(p)$, which stands, since for $0 < p \leq 1/2$,

$$Q(p) = \frac{p^2}{2}\Big(\frac{1-p}{p}\Big)^{2p} \geq \frac{p^2}{2}$$

and for $1/2 \le p < 1$,

$$Q(p) = Q(1-p) \ge \frac{(1-p)^2}{2}\,.$$

We finally show that in some cases our inequality (2.1) is better and in other cases is worse than inequality (1.2). For example, for $p = 0.4$ and $x = 10$,

$$\begin{aligned} 1.557 \approx \Big(\frac{11^2}{40}\Big)^{0.4} &= K^{0.4}(10) < 1 + Q(0.4)\log^2 10 \\ &= 1 + \frac{0.4^2}{2}\Big(\frac{0.6}{0.4}\Big)^{0.8}\log^2 10 \approx 1.586 \end{aligned}$$

and for $p = 0.4$ and $x = 30$,

$$2.298 \approx \Big(\frac{31^2}{120}\Big)^{0.4} = K^{0.4}(30) > 1 + Q(0.4)\log^2 30 \approx 2.280.$$

3. An application

Let A and B be two positive invertible operators. Let us define as usual the weighted arithmetic mean as

$$A \nabla_p B := (1-p)A + pB,$$

the weighted geometric mean as

$$A \sharp_p B := A^{1/2}\big(A^{-1/2}BA^{-1/2}\big)^p A^{1/2}$$

and the relative operator entropy as

$$S(A \mid B) := A^{1/2}\log(A^{-1/2}BA^{-1/2})A^{1/2}.$$

Then we have an operator inequality version of (2.1):

$$A \sharp_p B + K^*(A \sharp_p B)K \le A \nabla_p B, \tag{3.1}$$

where $K = \sqrt{Q(p)}\,A^{-1}S(A \mid B)$ and $Q(p)$ is from (2.1).

Proof of (3.1). From (2.1) we get

$$\sqrt{a} + Q(p)\log(a)\,a^p\log(a) \le pa + (1-p),$$

whence for $X = A^{-1/2}BA^{-1/2}$,

$$X^p + Q(p)\log(X)\,X^p\log(X) \le pX + (1-p)I.$$

Multiplying $A^{1/2}$ to the above inequality from the left hand side and the right hand side, we obtain

$$A \sharp_p B + Q(p) A^{1/2} \log (A^{-1/2} B A^{-1/2}) A^{-1/2} (A \sharp_p B) A^{-1/2} \\ \times \log (A^{-1/2} B A^{-1/2}) A^{1/2} \le pB + (1-p)A,$$

so we have got (3.1). $\square$

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